this implies that $\varphi = 0$, the latter defining the invariant manifold Γ_1^1.

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ON CERTAIN DIMENSION PROPERTIES OF A CONTROL STABILIZING A MECHANICAL SYSTEM

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L. K. LILOV
(Sofia)
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The problem of determining the smallest number of controls stabilizing the equilibrium position of a mechanical system is investigated. Necessary and sufficient conditions are established under which stabilization of the equilibrium position is possible with a control of minimal dimension, and this dimension is determined. The influence of gyroscopic and dissipative forces on the dimension of the stabilizing control is studied completely for a linear approximation of the system being considered. Necessary conditions are found under which stabilization is possible by forces which depend only on the velocity.

1. We consider a controlled conservative mechanical system with n degrees of freedom, whose motion is described by the Lagrange equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial \Pi}{\partial q_i} = Q_i(u_1, \ldots, u_r), \quad Q_i(0, \ldots, 0) = 0 \quad (i = 1, \ldots, n) \qquad (1.1)$$

Here q represents the generalized coordinates, T and Π are the specified kinetic and potential energies, respectively, $u_1, \ldots, u_r$ are the controls. The functions $Q_i(u_1, \ldots, \ldots, u_r)$ are to be determined.

Suppose that system (1.1) has an equilibrium position $q = q^\circ$ when $u \equiv 0$. Without loss of generality we can take $q^\circ = 0$. We pose the problem of determining the smallest number of controls by means of which we can stabilize upto asymptotic stability the trivial solution $q = q^{\circ\cdot} = 0$ of system (1.1) for a certain choice of Q. If r is this number and $Q_i^\circ(u_1, \ldots, u_r)$ are the corresponding functions, then there exists an r-dimensional control $u = u^\circ(q, q^\cdot)$ stabilizing the solution $q = q^\cdot = 0$ of system (1.1) when $Q_i = Q_i^\circ$, and it is impossible to find functions of less than r variables, say, $Q_i(u_1, \ldots, u_{r_1})$ $(r_1 < r)$, for which it is possible to choose controls $u_1(q, q^\cdot), \ldots, \ldots, u_{r_1}(q, q^\cdot)$ stabilizing the equilibrium position $q = q^\cdot = 0$.

In a neighborhood of the equilibrium position $q = q^\cdot = 0$ the kinetic and potential energies can be represented in the form

$$T = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}q_i^\cdot q_j^\cdot + (**), \quad a_{ij} = a_{ji} = \text{const}$$

$$\Pi = -\frac{1}{2}\sum_{i,j=1}^{n} c_{ij}q_i q_j + (**), \quad c_{ij} = c_{ji} = \text{const}$$

where $(**)$ denotes a sum of terms of the third and higher orders in q_i and $q_i^\cdot$ $(i = 1, \ldots, \ldots, n)$.

The first approximation of system (1.1) is written in the form

$$Aq^{\cdot\cdot} = Cq + Pu, \quad A = \|a_{ij}\|, \quad C = \|c_{ij}\| \qquad (i, j = 1, \ldots, n) \tag{1.2}$$

Here P is an $(n \times r)$-matrix to be determined.

By λ_i we denote the roots of the equation $\det\|C - \lambda A\| = 0$ and by f_i the corresponding eigenvectors, $Cf_i = \lambda_i A f_i$. We make the change of variables $q = \Phi y$, $\Phi = \|f_1, \ldots, f_n\|$. Then, system (1.2) is reduced to the form [1]

$$y_i^{\cdot\cdot} = \lambda_i y_i + (\Phi^* P u)_i \qquad (i = 1, \ldots, n)$$

Here and subsequently the asterisk denotes transposition. We set

$$y_i = x_{2i}, \quad y_i^\cdot = x_{2i-1} \qquad (i = 1, \ldots, n)$$

$$x^{(1)*} = \|x_1, x_3, \ldots, x_{2n-1}\|, \quad x^{(2)*} = \|x_2, x_4, \ldots, x_{2n}\|, \quad x^* = \|x^{(1)*}, x^{(2)*}\|$$

$$\Lambda = \begin{Vmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{Vmatrix}, \quad L = \begin{Vmatrix} 0 & \Lambda \\ E & 0 \end{Vmatrix}, \quad B^* = \|(\Phi^* P)^*, 0\| \tag{1.3}$$

E is the unit $n \times n$ matrix

Then system (1.2) takes the form

$$x^{(1)\cdot} = \Lambda x^{(2)} + \Phi^* P u, \quad x^{(2)\cdot} = E x^{(1)}, \quad x^\cdot = Lx + Bu \tag{1.4}$$

Suppose that we are given a system of the general form

$$z^\cdot = Fz + Gu + f(z) + g(u) \tag{1.5}$$

Here z is an m-dimensional vector, F, G are matrices, f, g are vector-valued functions whose expansions start with terms of the second order of smallness.

The following assertion is valid [2]. In order that it be possible to stabilize the solution $z = 0$ of the linear approximation of system (1.5) upto asymptotic stability by an r-dimensional control for a certain choice of the matrix G, and that it be impossible to make such a stabilization by an $(r - 1)$-dimensional control for any choice of G whatsoever, it is necessary and sufficient that all the roots of the largest common divisor $D_{m-r}(\lambda)$ of the $(m - r)$th-order minors of the matrix $\| F - \lambda E \|$ have negative real parts, or that $D_{m-r}(\lambda) \equiv 1$ and $D_{m-r+1}(\lambda)$ has a root with a nonnegative real part. The condition that the real parts of all roots of $D_{m-r}(\lambda)$ be negative or that $D_{m-r}(\lambda) \equiv 1$ is sufficient for the choosing of the r-dimensional control u for the complete system (1.5) with some matrix G and with $g \equiv 0$.

If we assume that in $D_{m-r+1}(\lambda)$ there is a root with positive real part, while in $D_{m-r}(\lambda)$ all the roots are located to the left of the imaginary axis or $D_{m-r}(\lambda) \equiv 1$, then there are no G and $g(u)$ whatsoever for which we can select an r_1-dimensional control $(r_1 < r)$ stabilizing the solution $z = 0$ of system (1.5) upto asymptotic stability, i.e. in this case r is the minimal possible dimension of the control.

If by $\psi_1(\lambda), \ldots, \psi_t(\lambda)$ we denote the invariant polynomials of matrix F, then, as is known from [1], the m-dimensional Euclidean space R_m can always be split up into subspaces $I_1, \ldots, I_t$, cyclic relative to the given linear operator F, with the minimal polynomials $\psi_1(\lambda), \ldots, \psi_t(\lambda)$. Then, if all the roots in $D_{m-r}(\lambda)$ have negative real parts, or if $D_{m-r}(\lambda) \equiv 1$, as the matrix G we can take the matrix $\| g_1, \ldots, g_r \|$, where g_i is the generating vector of I_i $(i = 1, \ldots, n)$.

Let us apply these results to the conservative system (1.1) being considered and to its linear approximation (1.2) or (1.4).

We assume that among the roots λ_i $(i = 1, \ldots, n)$ there are p distinct ones. We set

$$\begin{gathered}\lambda_1 = \lambda_2 = \ldots = \lambda_{s_1}, \quad \lambda_{s_1+1} = \lambda_{s_1+2} = \ldots = \lambda_{s_1+s_2}, \ldots \\ \ldots, \lambda_{s_1+\ldots+s_{p-1}+1} = \ldots = \lambda_{s_1+\ldots+s_p} \quad (s_1 + \ldots + s_p = n, \\ \lambda_{s_1} \neq \lambda_{s_1+s_2} \neq \ldots \neq \lambda_{s_1+\ldots+s_p})\end{gathered} \tag{1.6}$$

Without loss of generality we can take

$$1 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_p \leqslant n \tag{1.7}$$

The following assertion is valid.

Theorem 1.1. Let $\lambda_{s_1+\ldots+s_p}$ be the root of highest multiplicity s_p of the equation $\det \| C - \lambda A \| = 0$. The trivial solution $q = q^{\cdot} = 0$ of systems (1.1) and (1.2) can always be made asymptotically stable by means of the s_p controls $u = \| u_1, \ldots, \ldots, u_{s_p} \|$. The number s_p will be the minimal possible for the linear system (1.2). If $s_p = s_{p-1} = \ldots = s_k$ $(k \leqslant p)$ and if even one of the numbers $\lambda_{s_1+\ldots+s_p}, \lambda_{s_1+\ldots+s_{p-1}}, \ldots, \lambda_{s_1+\ldots+s_k}$ is positive, then the number s_p will be the minimal possible also for the complete system (1.1).

Proof. With the aid of elementary transformations the characteristic matrix $\|L - \lambda E_{2n}\|$ (E_{2n} is the $2n \times 2n$ unit matrix) of system (1.4) can be brought to the equivalent diagonal matrix $\|l_{ij}\|$ $(i, j = 1, \ldots, 2n)$

$$l_{ij} = 0 \quad (i \neq j), \qquad l_{11} = l_{22} = \ldots = l_{2n-s_p,\, 2n-s_p} = 1$$

$$l_{2n-s_p+1,2n-s_p+1} = \ldots = l_{2n-s_{p-1},\,2n-s_{p-1}} = \lambda^2 - \lambda_{s_1+\ldots+s_p}$$

$$l_{2n-s_{p-1}+1,\,2n-s_{p-1}+1} = \ldots = l_{2n-s_{p-2},\,2n-s_{p-2}} = (\lambda^2 - \lambda_{s_1+\ldots+s_{p-1}})(\lambda^2 - \lambda_{s_1+\ldots+s_p})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$l_{2n-s_1+1,\,2n-s_1+1} = \ldots = l_{2n,\,2n} = (\lambda^2 - \lambda_{s_1})(\lambda^2 - \lambda_{s_1+s_2})\ldots(\lambda^2 - \lambda_{s_1+\ldots+s_p})$$

From the form of the matrix l_{ij} it is obvious that the largest common divisor of the $(2n - s_p)$th-order minors is $D_{2n-s_p}(\lambda) \equiv 1$, while

$$D_{2n-s_p+1}(\lambda) = (\lambda^2 - \lambda_{s_1+\ldots+s_p})$$

i.e. $D_{2n-s_p+1}(\lambda)$ has a root with a nonnegative real part. According to [2] this is necessary and sufficient for it to be possible to stabilize the solution $x = 0$ of system (1.4) upto asymptotic stability by an s_p-dimensional control for some choice of matrix B and impossible to stabilize it asymptotically by a k-dimensional control ($k < s_p$) for any choice of B whatsoever. To prove this assertion also for system (1.2) we need to show only that the matrix B in (1.4) can be taken in the form (1.3).

Indeed, let $\psi_i(\lambda)$ be the ith polynomial of matrix L. Since λ is contained in $\psi_i(\lambda)$ only in the form λ^2 and the argument

$$L^2 = \begin{Vmatrix} \Lambda & 0 \\ 0 & \Lambda \end{Vmatrix}$$

we have

$$\psi_i(L) = \begin{Vmatrix} \psi_i^{(1)}(\Lambda) & 0 \\ 0 & \psi_i^{(1)}(\Lambda) \end{Vmatrix} \tag{1.8}$$

Here $\psi_i^{(1)}(\lambda)$ is the ith invariant polynomial of the matrix Λ, equalling, obviously,

$$\psi_i^{(1)}(\lambda) = \psi_i(\sqrt{\lambda})$$

If m_i is the degree of polynomial $\psi_i(\lambda)$, then, clearly, the degree of $\psi_i^{(1)}(\lambda)$ is $^1/_2\, m_i$. By I_i we denote a cyclic subspace of the $2n$-dimensional Euclidean space R_{2n}, with characteristic polynomial $\psi_i(\lambda)$, and by $I_i^{(1)}$ a cyclic subspace of the n-dimensional Euclidean space R_n, with characteristic polynomial $\psi_i^{(1)}(\lambda)$. Let $b_i^{(1)}$ be the generating vector of $I_i^{(1)}$. Then by the definition of a generating vector.

$$\psi_i^{(1)}(\Lambda)\, b_i^{(1)} = 0, \qquad \det \| b_i^{(1)},\ \Lambda b_i^{(1)}, \ldots, \Lambda^{^1/_2 m_i - 1} b_i^{(1)} \| \neq 0 \tag{1.9}$$

Let us show that as the generating vector of I_i we can take the $2n$-dimensional vector b_i, $b_i^* = \| b_i^{(1)*},\ 0 \|$. Indeed, from (1.8) and (1.9) follows

$$(\psi_i(L)\, b_i)^* = \| [\psi_i^{(1)}(\Lambda)\, b_i^{(1)}]^*,\ \ 0 \| = \| 0, 0 \|$$

On the other hand, the vectors b_i, Lb_i, ..., $L^{m_i-1} b_i$ are linearly independent, which is obvious from (1.9) and from the form of the matrix

$$\| b_i,\ Lb_i,\ \ldots,\ L^{m_i-1} b_i \|$$

which differs only in the order of the columns from the matrix

$$\begin{Vmatrix} b_i^{(1)} & \Lambda b_i^{(1)} & \ldots & \Lambda^{^1/_2 m_i - 1} b_i^{(1)} & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & b_i^{(1)} & \Lambda b_i^{(1)} & \ldots & \Lambda^{^1/_2 m_i - 1} b_i^{(1)} \end{Vmatrix}$$

After the matrix $B^{(1)} = \| b_1^{(1)}, \ldots, b_r^{(1)} \|$ is found, the matrix P in (1.2) is determined from the formula $P = (\Phi^*)^{-1}B^{(1)}$.

Completely analogously we can show that if zero values are absent among the eigenvalues λ_i $(i = 1, \ldots, n)$, the trivial solution of system (1.2) can be stabilized upto asymptotic stability by forces dependent only on the velocity $q^{\cdot}$ and the acceleration $q^{\cdot\cdot}$. For this it is enough to take the matrix B in (1.4) in the form $B^* = \| 0, B^{(1)*} \|$, to determine the stabilizing control $u = Mx$ (M is some $(r \times 2n)$-matrix), and to pass from system (1.4) to system (1.2). The remaining assertions of Theorem 1.1 follow in obvious fashion from the above-mentioned results [2].

From Theorem 1.1 it follows that the solution $q = q^{\cdot} = 0$ of system (1.2) can be stabilized asymptotically by one control if and only if $s_p = 1$. In accordance with (1.7) this condition is equivalent to the conditions

$$\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_n \tag{1.10}$$

If we assume additionally that the controls behave in a specific manner, i. e. if we impose constraints on the vector p to which the matrix P in (1.2) reduces in the case being considered, then for the stabilizability of the trivial solution of system (1.2) we require certain other conditions besides conditions (1.10). For example, if $p^* = \| 1, 0, \ldots, 0 \|$, then, as was shown in [3], besides the fulfillment of (1.10) it is further necessary that all the elements of the first column of matrix Φ^* be nonzero.

2. The control $u = \| u_1, \ldots, u_r \|$, stabilizing the trivial solution of system (1.1) upto asymptotic stability, depends, in general, on the velocity as well as on the position. Let us investigate the conditions under which it is possible to choose the controls as functions of velocity alone. The following assertion is valid.

Theorem 2.1. If the equilibrium position $q = q^{\cdot} = 0$ of system (1.2) can be made asymptotically stable by a one-dimensional control dependent linearly only on the velocity for some choice of matrix P, then this equilibrium position is necessarily stable and all the roots of the equation $\det \| C - \lambda A \| = 0$ are negative.

Proof. According to Theorem 1.1, since the solution $q = q^{\cdot} = 0$ of system (1.2), or, equally, the solution $x^{(1)} = x^{(2)} = 0$ of system (1.4), can be stabilized by one control, it is necessary that conditions (1.10) be fulfilled and that there exists a vector $b^* = \| b^{(1)*}, 0 \|$, for which it is possible to choose a stabilizing control in (1.4) in the form $u = u(x^{(1)})$. The vectors $b^{(1)}, \Lambda b^{(1)}, \ldots, \Lambda^{n-1} b^{(1)}$ are linearly independent, otherwise asymptotic stabilization by one control is impossible.

By M_1 we denote the matrix

$$M_1 = \| b^{(1)}, \ \Lambda b^{(1)}, \ldots, \Lambda^{n-1} b^{(1)} \|$$

and in (1.4) we make the change of variables

$$x = Mz, \quad M = \left\| \begin{matrix} M_1 & 0 \\ 0 & M_1 \end{matrix} \right\|$$

then we can write (1.4) in the form

$$z^{(1)\cdot} = M_1^{-1} \Lambda M_1 z^{(2)} + eu$$
$$z^{(2)\cdot} = E z^{(1)}, \qquad z^{\cdot} = M^{-1} L M z + M^{-1} bu \tag{2.1}$$

Here

$$z^{(1)} = M_1^{-1} x^{(1)}, \qquad z^{(2)} = M_1^{-1} x^{(2)}$$
$$e^* = \| 1, 0, \ldots, 0 \|, \qquad z^* = \| z^{(1)*}, \ z^{(2)*} \|$$

Since by assumption the control

$$u = \mu^* z^{(1)}, \qquad \mu^* = \| \mu_1, \ldots, \mu_n \|$$

asymptotically stabilizes system (2.1), the characteristic equation of system (2.1) with $u = \mu^* z^{(1)}$ must have all roots with negative real parts. As is easily seen, this characteristic equation reduces to the form

$$(-1)^n \det \| \Lambda - \lambda^2 E \| + \chi(\lambda) = 0 \tag{2.2}$$

Here $\chi(\lambda)$ is the determinant of the matrix which is obtained from the matrix $\| M_1^{-1} \Lambda M_1 - \lambda^2 E \|$ by replacing the first row by the row

$$\| \mu_1 \lambda, \ \mu_2 \lambda, \ldots, \mu_n \lambda \|$$

Polynomial (2.2) is Hurwitz polynomial by assumption and, therefore, all of its coefficients are necessarily positive [1]. But $\chi(\lambda)$ contains only the odd powers of λ and, consequently, all the coefficients of the polynomial $(-1)^n \det \| \Lambda - \lambda^2 E \|$ are positive. However, this is possible only if $\lambda_i < 0$, $(i = 1, \ldots, n)$.

The necessity of the conditions $\lambda_i < 0$ $(i = 1, \ldots, n)$ for it to be possible to stabilize the solution $q = q^{\cdot} = 0$ of system (1.2) by forces dependent only on the velocity was proven in [5] under the assumption that the minimal value of the quadratic optimizing functional J, namely, $J_0 = \min J$, considered as a function of the initial state q_0, $q_0^{\cdot}$ of the system, is represented as a sum of two terms, one of which depends only on the position q_0, while the other, only on the velocity $q_0^{\cdot}$. As we see from Theorem 2.1, this assumption is unnecessary.

The conditions $\lambda_i < 0$ $(i = 1, \ldots, n)$ is not only necessary but also sufficient for the asymptotic stabilization of the solution $q = q^{\cdot} = 0$ of system (1.1) and (1.2) by forces depending only on the velocity. It turns out here that as such forces we can always take dissipative forces [5, 6].

3. Let us assume that additional dissipative or gyroscopic forces can be imposed on system (1.1). The question of the influence of dissipative and gyroscopic forces on the controllability of system (1.2) was studied in [5, 7, 8]. In this and the following sections this influence is examined from another point of view, namely: having added dissipative or gyroscopic forces to system (1.2), to what extent can we succeed in lowering the dimension of the control sufficient for the asymptotic stabilization of the trivial solution of (1.2).

Theorem 3.1. 1°. If among the roots (1.6) of the equation $\det \| C - \lambda A \| = 0$ there is a zero root $\lambda_{s_1 + \ldots + s_k}$ of multiplicity s_k, then, having added suitably selected dissipative forces to system (1.2), we can always achieve the asymptotic stability of the solution $q = q^{\cdot} = 0$ of system (1.2) by an s_k-dimensional control under a suitable choice of matrix P. The number s_k is the minimal possible, i.e. for any dissipative forces added to the system (1.2) it is impossible to find an r-dimensional control $(r < s_k)$ which asymptotically stabilizes the solution $q = q^{\cdot} = 0$ of system (1.2) for any choice whatsoever of an $(n \times r)$-matrix P.

2°. If there are no zero roots among the roots (1.6) of the equation $\det \| C - \lambda A \| = 0$, then, having added suitably selected dissipative forces to system (1.2), we can always achieve the asymptotic stability of the solution $q = q^{\cdot} = 0$ of system (1.2) by a one-dimensional control under a suitable choice of an $(n \times 1)$-matrix P.

3°. If the hypothesis in 2° is fulfilled and if all the roots of the equation $\det \| C - \lambda A \| = 0$ are negative, then the solution $q = q^{\cdot} = 0$ of the system (1.2) can be made asymptotically stable by the addition of dissipative forces [4].

If the zero root $\lambda_{s_1+\ldots+s_k}$ is the root of highest multiplicity, i.e. $s_k = s_p$, then, as follows from the Theorem 1.1 and from item 1° of Theorem 3.1, the addition of dissipative forces does not lower the dimension of the stabilizing control.

Proof. Let us first consider the influence of dissipative forces. Let $D = \|d_{ij}\|$, $d_{ij} = d_{ji}$ $(i, j = 1, \ldots, n)$, be a positive definite matrix. We consider the system

$$x^{(1)\cdot} = -Dx^{(1)} + \Lambda x^{(2)} + B^{(1)}u, \qquad x^{(2)\cdot} = Ex^{(1)} \tag{3.1}$$

System (3.1) differs from system (1.4) in the presence of dissipative forces with the Rayleigh function

$$\frac{1}{2}\sum_{i,j=1}^{n} d_{ij}x_{2i-1}x_{2j-1}$$

By elementary transformations we can reduce the characteristic matrix of system (3.1) with $u \equiv 0$ to the form

$$\begin{Vmatrix} E & 0 \\ 0 & \lambda^2 E + \lambda D - \Lambda \end{Vmatrix} \tag{3.2}$$

Here E is the unit matrix.

In what follows we shall assume that $d_{ij} = 0$ when $i \neq j$. Then, obviously, matrix (3.2) is diagonal. To any root $\lambda_{s_1+\ldots+s_q}$ from (1.6) there correspond s_q elements in (3.2),

$$\lambda^2 + d_{ii}\lambda - \lambda_{s_1+\ldots+s_q} \qquad (i = s_1 + \ldots + s_{q-1} + 1, \ldots, s_1 + \ldots + s_q) \tag{3.3}$$

Let us require that the polynomials (3.3) be relatively prime. If $\lambda_{s_1+\ldots+s_q} \neq 0$ it is obvious that this can always be achieved by a suitable choice of $d_{ii} > 0$. It is enough to require that

$$d_{ii} \neq d_{jj} \qquad (i \neq j) \qquad (i = s_1 + \ldots + s_{q-1} + 1, \ldots, s_1 + \ldots + s_q) \tag{3.4}$$

If $\lambda_{s_1+\ldots+s_q} = 0$, the polynomials (3.3) contain a common factor λ for arbitrary values of d_{ii} and may not be relatively prime. Any two polynomials (3.3) corresponding to distinct roots can be made relatively prime by a suitable choice of coefficients d. For example, let us assume that $\lambda_1 \neq \lambda_2$ and let us consider the polynomials

$$\lambda^2 + d_{11}\lambda - \lambda_1, \quad \lambda^2 + d_{22}\lambda - \lambda_2$$

They will be relatively prime if their resultant [9]

$$(\lambda_1 - \lambda_2)^2 + (d_{11} - d_{22})(\lambda_1 d_{22} - \lambda_2 d_{11}) \neq 0 \tag{3.5}$$

which can always be achieved.

Suppose that one of the roots (1.6) is zero, say, $\lambda_{s_1+\ldots+s_k} = 0$. Then, in accord with what was said above, by choosing $d_{ii} > 0$ in such a way that conditions (3.4) and (3.5) are fulfilled, we can reduce matrix (3.2) to the equivalent diagonal matrix

$$\|l_{ij}\| \qquad (i, j = 1, \ldots, 2n)$$

$$l_{ij} = 0 \quad (i \neq j), \qquad l_{11} = l_{22} = \ldots = l_{2n-s_k,\, 2n-s_k} = 1$$

$$l_{2n-s_k+1,\, 2n-s_k+1} = (\lambda^2 + d_{\sigma_k+1,\, \sigma_k+1}\lambda) \prod_{\substack{\alpha=1 \\ \alpha\neq k}}^{p} \prod_{i=\sigma_\alpha+1}^{\sigma_\alpha+s_\alpha} (\lambda^2 + d_{ii}\lambda - \lambda_{s_1+\ldots+s_\alpha})$$

. . . .

$$l_{2n,\, 2n} = (\lambda^2 + d_{\sigma_k+s_k,\, \sigma_k+s_k}\lambda) \prod_{\substack{\alpha=1 \\ \alpha\neq k}}^{p} \prod_{i=\sigma_\alpha+1}^{\sigma_\alpha+s_\alpha} (\lambda^2 + d_{ii}\lambda - \lambda_{s_1+\ldots+s_\alpha})$$

$$\sigma_k = s_1 + \ldots + s_{k-1}$$

Obviously, $D_{2n-s_k}(\lambda) \equiv 1$ and the trivial solution of system (3.1) can, according to [2], be stabilized by an s_k-dimensional control for a suitable choice of the matrix $B^{(1)}$. Thus, the addition of dissipative forces of a particular form $(d_{ij} = 0,\ i \neq j)$ to system (1.4) or, what is the same, to system (1.2) permits us to lower the number of controls from s_p to s_k, where, in accord with (1.7), $s_k \leqslant s_p$.

On the other hand, whatever be the forces, linearly dependent on velocity alone, added to (1.2), stabilization by an r-dimensional control $(r < s_k)$ is impossible. Let us assume to the contrary that there does exist an $(n \times r)$-matrix P for which stabilization by an r-dimensional control is possible. Let us consider the group of equations from (1.2) corresponding to the zero root $\lambda_{s_1+\ldots+s_k} = 0$. Since the number s_k of equations in the group is larger than the number r of controls $u_1, \ldots, u_r$, the latter can always be eliminated and, whatever be the added forces, linearly dependent only on the velocity, we can obtain at least one integral in the form $f(q, q^\cdot) = \text{const}$, not depending on u. But the presence of this integral signifies that the solution $q = q^\cdot = 0$ cannot be stabilized by any control $u = \| u_1, \ldots, u_r \|$ whatsoever.

If there are no zero roots among the roots (1.6), then matrix (3.2) can be reduced to the matrix

$$\| l_{ij} \| \quad (i, j = 1, \ldots, 2n)$$

$$l_{ij} = 0 \quad (i \neq j), \qquad (l_{11} = l_{22} = \ldots = l_{2n-1,\, 2n-1} = 1$$

$$l_{2n,\, 2n} = \prod_{\alpha=1}^{p} \prod_{i=s'}^{s''} (\lambda^2 + d_{ii}\lambda - \lambda_{s_1 \ldots + s_\alpha}) \qquad \begin{pmatrix} s'' = s_1 + \cdots + s_\alpha \\ s' = s_1 + \cdots + s_{\alpha-1} + 1 \end{pmatrix}$$

Obviously, $D_{2n-1}(\lambda) \equiv 1$ and in this case system (3.1) can be made asymptotically stable by one control or is already asymptotically stable if all $\lambda_i < 0$ $(i = 1, \ldots, n)$. Indeed, in the latter case all the roots of the characteristic polynomial

$$D_{2n}(\lambda) = \prod_{\alpha=1}^{p} \prod_{i=s'}^{s''} (\lambda^2 + d_{ii}\lambda - \lambda_{s_1+\cdots+s_\alpha}) \quad \begin{pmatrix} s'' = s_1 + \cdots + s_\alpha) \\ s' = s_1 + \cdots + s_{\alpha-1} + 1 \end{pmatrix}$$

of matrix (3.2) have negative real parts. In other words, in this particular case we obtain the known result [4] that an isolated and stable equilibrium position of system (1.2) can be made asymptotically stable by dissipative forces.

We have thus proven the validity of Theorem 3.1.

4. We now assume that gyroscopic forces are added to system (1.2).

Theorem 4.1. 1°. If among the roots (1.6) of the equation $\det \| C - \lambda A \| = 0$ there is a zero root $\lambda_{s_1+\ldots+s_k} = 0$ of multiplicity s_k, then, having added suitably selected gyroscopic forces to system (1.2), we can always achieve the asymptotic stability of the solution $q = q^\cdot = 0$ of system (1.2) by an s_k-dimensional control $u = u(q, q^\cdot)$ under a suitable choice of matrix P. The number s_k is the minimal possible, i.e. for any gyroscopic forces added to the system (1.2) it is impossible to find an r-dimensional control $(r < s_k)$ which asymptotically stabilizes the solution $q = q^\cdot = 0$ of system (1.2) for any choice whatsoever of matrix P.

2°. If there are no zero roots among the roots (1.6) of the equation $\det \| C - \lambda A \| = 0$, then, having added suitably selected gyroscopic forces to system (1.2), we can always achieve the asymptotic stability of the solution $q = q^\cdot = 0$ of system (1.2) by a one-dimensional control under a suitable choice of an $(n \times 1)$-matrix P.

Proof. Let $H = \| h_{ij} \|$, $h_{ij} = -h_{ji}$ $(i, j = 1, \ldots, n)$ be some matrix. Let us consider the system

$$x^{(1)\cdot} = -Hx^{(1)} + \Lambda x^{(2)} + B^{(1)}u, \qquad x^{(2)\cdot} = Ex^{(1)} \tag{4.1}$$

obtained from system (1.4) by the addition of gyroscopic forces. The characteristic matrix of system (4.1) is reduced to the equivalent matrix

$$\left\| \begin{matrix} E & 0 \\ 0 & \lambda^2 E + \lambda H - \Lambda \end{matrix} \right\|, \quad E \text{ is the unit matrix,} \tag{4.2}$$

We shall take it that

$$h_{i,\, i+1} \neq 0, \quad h_{ik} = 0 \qquad (k = i + 2, \ldots, n \quad i = 1, 2, \ldots, n) \tag{4.3}$$

Then the matrix $\lambda^2 E + \lambda H - \Lambda$ has the form

$$\left\| \begin{matrix} \lambda^2 - \lambda_1 & h_{12}\lambda & 0 & \ldots & 0 \\ -h_{12}\lambda & \lambda^2 - \lambda_2 & h_{23}\lambda & \ldots & 0 \\ 0 & -h_{23}\lambda & \lambda^2 - \lambda_3 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & \lambda^2 - \lambda_n \end{matrix} \right\| \tag{4.4}$$

Among the roots λ_i ($i = 1, \ldots, n$) from (1.6) there are p distinct ones. If one of them is zero, without loss of generality we can assume that the corresponding group of elements is located in the lower right corner of matrix (4.4).

Let $\lambda_1 \neq 0$. We multiply the second row of matrix (4.4) by $-\lambda / h_{12}$ ($h_{12} \neq 0$ according to (4.3)) and we add it to the first. We multiply the first column of the new matrix obtained by $h_{12}\lambda / \lambda_1$ and we add it to the second. Further, by subtracting the first row of the new matrix, multiplied by an appropriate polynomial, from the second and third rows, we arrive at the matrix

$$\left\| \begin{matrix} 1 & 0 & 0 & \ldots 0 \\ 0 & \varphi_1(\lambda) - \lambda_2 & h_{23}\lambda & \ldots 0 \\ 0 & \varphi_2(\lambda) & \lambda^2 - \lambda_3 & \ldots 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 \end{matrix} \right\|$$

Here $\varphi_1(\lambda)$, $\varphi_2(\lambda)$ are certain polynomials and $\varphi_1(0) = \varphi_2(0) = 0$. If $\lambda_2 \neq 0$, we can continue the indicated procedure, etc. Thus, if one of the roots (1.6), say, $\lambda_{s_1 + \ldots + s_k}$ of multiplicity s_k is zero, then matrix (4.2) can be reduced to the equivalent matrix

$$\| l_{ij} \| \quad (i, j = 1, \ldots, 2n)$$

$$l_{ij} = 0 \quad (i \neq j) \quad (i, j = 1, \ldots, 2n - s_k)$$

$$l_{11} = l_{22} = \cdots = l_{2n-s_k,\, 2n-s_k} = 1$$

$$l_{ij} = \varphi_{ij}(\lambda) \quad (i, j = 2n - s_k + 1, \ldots, 2n)$$

Here $\varphi_{ij}(\lambda)$ are certain polynomials, $\varphi_{ij}(0) = 0$. Obviously, in this case

$$D_{2n-s_k}(\lambda) \equiv 1$$

If there are no zero roots among the roots (1.6), then matrix (4.2) reduces to the matrix

$$\| l_{ij} \| \quad (i, j = 1, \ldots, 2n)$$

$$l_{ij} = 0 \quad (i \neq j), \quad l_{11} = l_{22} = \cdots = l_{2n-1,\, 2n-1} = 1, \quad l_{2n,\, 2n} = \varphi(\lambda)$$

Here $\varphi(\lambda)$ is a certain polynomial, $\varphi(0) = 0$.

In this case

$$D_{2n-1}(\lambda) \equiv 1$$

In precisely the same way as in Sect. 3, Theorem 4.1 is established from what is shown above.

Just as in Sect. 3 we can show that if the zero root $\lambda_{s_1+\ldots+s_k} = 0$ is the root of highest multiplicity, i.e. $s_k = s_p$, then the addition of gyroscopic forces does not allow us to lower the dimension of the stabilizing control.

Note. By the addition of dissipative or gyroscopic forces we can lower the dimension of the control which stabilizes the trivial solution of system (1.1) upto asymptotic stability from s_p to m (m is one of the numbers $0, 1, s_k$). This assertion is obvious from Theorems 3.1, 4.1 and from the known fact that the asymptotic stability of the trivial solution of the complete system (1.1) follows from the asymptotic stability of the trivial solution of the linear approximation (1.2) [10].

A comparison of Theorems 3.1 and 4.1 shows that gyroscopic and dissipative forces may in almost like fashion lower the dimension of the stabilizing control. The single case when dissipative forces can achieve more is the particular case of $\lambda_i < 0$ ($i = 1, \ldots, \ldots, n$) in which the addition of dissipative forces to system (1.2) can strengthen a stable equilibrium position $q = q' = 0$ upto asymptotic stability. However, in a number of cases it has proved to be preferable to add gyroscopic forces for lowering the dimension of the control because the addition of gyroscopic forces does not call for an additional expenditure of energy on the system's motion.

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